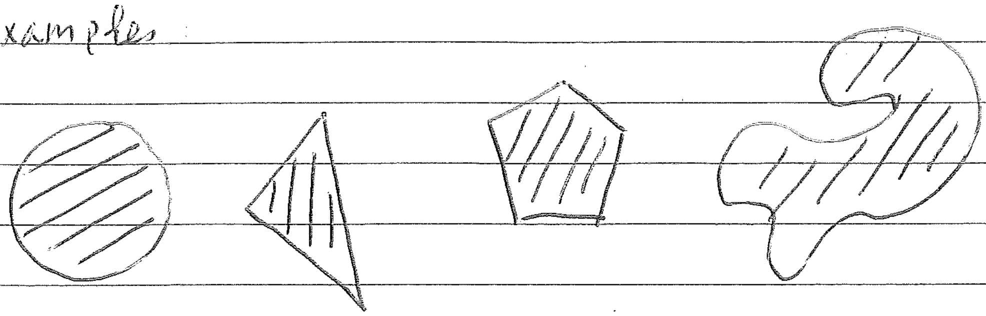
Week 2 (Jan 19)

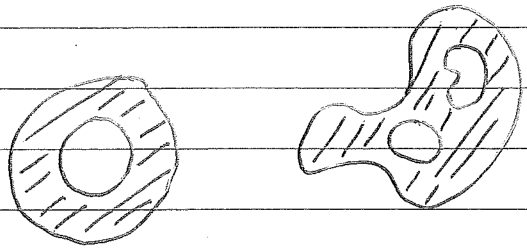
- Double Integral over regions
- Fubini's theorem for regions.

A region is the set bounded by one or several curves

Examples



bdd by a single curve



bdd by more than one curves

Let f be a function defined in a region D (a region is also called a domain), We extend f to be a function in $\mathbb{R}^2$ by setting

$$\tilde{f}(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D. \end{cases}$$

Define

$$\iint_D f \, dA = \iint_R \tilde{f} \, dA, \text{ where}$$

R is any rectangle containing D. As $\tilde{f} = 0$ outside D, clearly

$$\iint_R \tilde{f} \, dA = \iint_{R_1} \tilde{f} \, dA$$

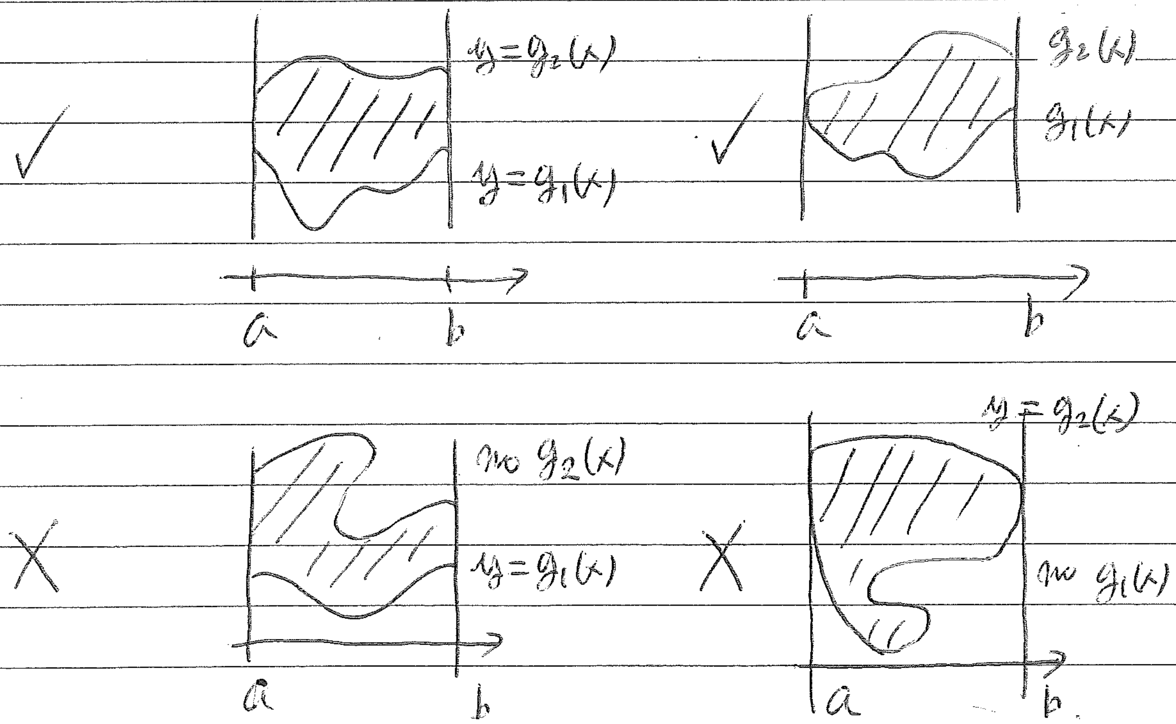
when R_1 is another rectangle containing D, so our definition is well-defined.

A remark: when f is continuous in D, $\tilde{f}$ may not be continuous in R since discontinuity may occur at the boundary of D. However, we point out that it is still piecewise continuous and Fubini's theorem can be applied to $\tilde{f}$ over R.

- Fubini's Theorem for Regions

In practice, we only need to consider special regions. Let's assume a region D can be described as

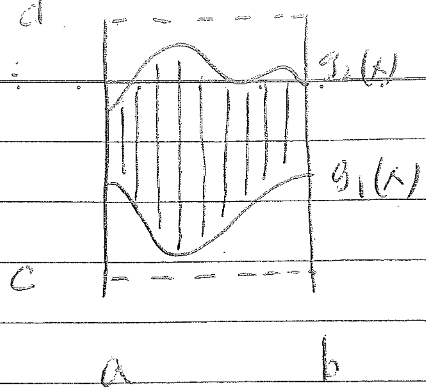
$$D = \{(x,y) : g_1(x) \le y \le g_2(x),\ a \le x \le b\} \qquad \boxed{\text{Type I region}}$$



Theorem Let D be described in the above manner where g_1, g_2 are continuous. Then

$$\iint_D f\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Pf:



Choose $R = [a,b] \times [c,d]$ to contain D

then

$$\iint_D f \, dA = \iint_R \tilde{f} \, dA$$

$$= \int_a^b \int_c^d \tilde{f}(x,y) \, dy \, dx \quad \text{(Fubini's}$$

For $x \in [a,b]$,
$$\int_c^d \tilde{f}(x,y) \, dy = \int_c^{g_1(x)} \tilde{f}(x,y) \, dy + \int_{g_1(x)}^{g_2(x)} \tilde{f}(x,y) \, dy + \int_{g_2(x)}^d \tilde{f}(x,y) \, dy$$

$$= \int_{g_1(x)}^{g_2(x)} \tilde{f}(x,y) \, dy \quad (\because \tilde{f} = 0 \text{ outside } D)$$

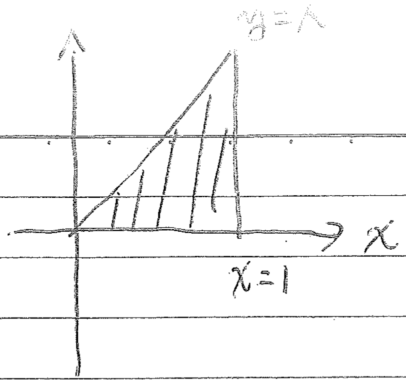
$$= \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \quad (\because \tilde{f} = f \text{ inside } D)$$

$$\therefore \int_a^b \int_c^d \tilde{f}(x,y) \, dy \, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx, \quad \text{DONE} \ \#$$

$\boxed{\text{e.g. 1}}$ Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis, the lines $y = x$ and $x = 1$, and whose top lies in the plane $z = 3 - x - y$.

the triangle D is described by

$$0 \le y \le x$$
$$0 \le x \le 1$$



the volume of the prism is

$$\iint_D (3-x-y)\,dA = \int_0^1 \int_0^x (3-x-y)\,dy\,dx \quad \text{(Fubini's)}$$

$$= \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx$$

$$= \left(\frac{3}{2}x^2 - \frac{x^3}{3} - \frac{x^3}{6}\right)\Bigg|_0^1$$

$$= 1. \ \#$$

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When x and y are exchanged, the region becomes

$$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\},$$ Type II region

and Fubini's theorem becomes

$$\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

$\boxed{\text{e.g. 1}}$ (cont'd) the triangle can be described in the second case: $y \le x \le 1$, $0 \le y \le 1$, so

$$\iint_D (3-x-y)\,dA = \int_0^1 \int_y^1 (3-x-y)\,dx\,dy$$

$$= \int_0^1 \left(3x - \frac{x^2}{2} - yx\right)\Bigg|_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy$$

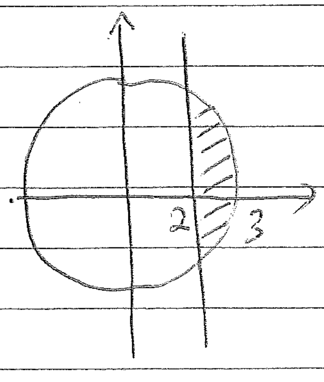
$$= \left(\frac{5}{2}y - 2y^2 + \frac{1}{2}y^3\right)\Bigg|_0^1$$

$$= \frac{5}{2} - 2 + \frac{1}{2}$$

$= 1$, same as before.

$\boxed{\text{e.g. 2}}$ Express $\iint_D f\,dA$ in $dy\,dx$ and $dx\,dy$, where

D is the region bounded by $x^2+y^2=9$ and $x=2$.



$x^2+y^2=9$, $x=2$ intersect at $y^2+4=9$, $y=\pm\sqrt{5}$

so, pts of intersection are $(2,\sqrt{5})$, $(2,-\sqrt{5})$.

D described as

$$-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, \quad 2 \le x \le 3$$

or $\quad 2 \le x \le \sqrt{9-y^2}, \quad -\sqrt{5} \le y \le \sqrt{5}$

So $$\iint_D f\,dA = \int_2^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{2}^{\sqrt{9-y^2}} f(x,y)\,dx\,dy.$$ Looks very different!

Sometimes, the order of integration matters.

$\boxed{\text{e.g. 3}}$ Evaluate $\iint_D \frac{\sin x}{x}\,dA$ where D is the triangle bounded by $y=x$, $x=1$, and the x-axis.

$$\iint_D \frac{\sin x}{x}\,dA = \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^1 \frac{\sin x}{x}\, y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \sin x\,dx$$ GOOD.

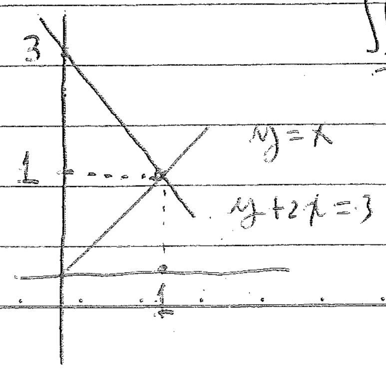
$$= -\cos 1 + 1 \quad \#$$

$$\iint_D \frac{\sin x}{x}\,dA = \int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy = \;??$$ NO WAY

Sometimes, $dy\,dx$ and $dx\,dy$ have preference.

$\boxed{\text{e.g. 4}}$ Let T be the triangle formed by $y=x$, $y+2x=3$ and the y-axis. Express

$\iint_T F\,dA$ in $dy\,dx$ and $dx\,dy$ respectively.



As a Type I-region, T is

$0 \le x \le 1$

$x \le y \le 3-2x$,

$$\iint_T f\,dA = \int_0^1 \int_x^{3-2x} f(x,y)\,dy\,dx$$

As a Type II-region, T is

$$0 \le y \le 3$$
$$0 \le x \le h_2(y), \text{ where}$$

$$h_2(y) = \begin{cases} y & 0 \le y \le 1 \\ \frac{1}{2}(3-y) & 1 \le y \le 3 \end{cases}$$

Hence

$$\iint_T f\,dA = \int_0^3 \int_0^{h_2(y)} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{h_2(y)} f(x,y)\,dx\,dy + \int_1^3 \int_0^{h_2(y)} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{\frac{1}{2}(3-y)} f(x,y)\,dx\,dy$$

The $dy\,dx$ approach is simpler than the $dx\,dy$ approach.

$\boxed{\text{e.g. 5}}$ Express $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$ in $dx\,dy$.

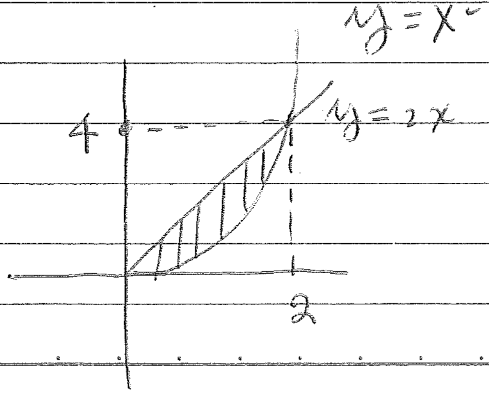
We need to recover the region first. Indeed, it is

$$0 \le x \le 2$$
$$x^2 \le y \le 2x.$$

Sketch the region:

point of intersection

Solve $\begin{cases} y = x^2 \\ y = 2x \end{cases}$ to get $x=2,\ y=4$ or $x=0,\ y=0$



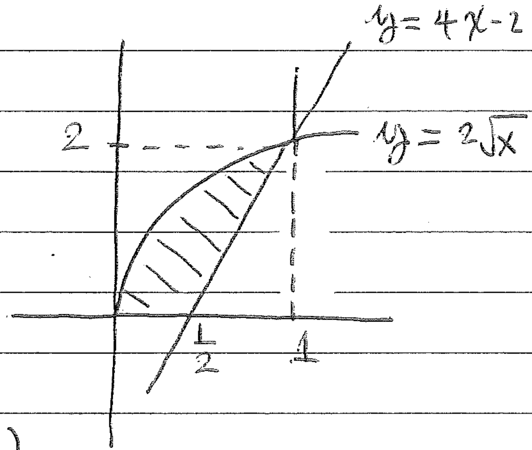
As a Type II-region,

$$0 \le y \le 4$$
$$\frac{y}{2} \le x \le \sqrt{y}$$

$$\therefore \int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy. \quad \#$$

$\boxed{\text{eg } 6}$ Find the volume of the wedgelike solid that lies beneath $z = 16 - x^2 - y^2$ and above R: $y = 2\sqrt{x}$, $y = 4x - 2$, and the x-axis.

First, sketch the region:



points of intersection:

$$\begin{cases} y = 4x - 2 \\ y = 2\sqrt{x} \end{cases}$$

$\Rightarrow y = 2$ or -1

they are $(1, 2)$ and $(1/4, -1)$.

R is not of Type I, but of Type II:

$$0 \le y \le 2$$
$$y^2/4 \le x \le \frac{1}{4}(y+2)$$

$\therefore$ The volume

$$= \iint_R (16 - x^2 - y^2)\, dA$$

$$= \int_0^2 \int_{y^2/4}^{\frac{1}{4}(y+2)} (16 - x^2 - y^2)\, dx\, dx$$

$$= \cdots = \frac{20803}{1680} \approx 12.4 \text{ (see Text)}$$

• Some theoretical aspects.

Fubini's theorem Let f be a piecewise continuous function on a rectangle R. Then

$$\iint_R f\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

$$= \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

$$\iint_R f\, dA \approx \sum_{j,k} f(P_{jk}) \Delta x_j \Delta y_k$$

$$= \sum_j \left(\sum_k f(P_{jk}) \Delta y_k \right) \Delta x_j$$

Choose $P_{jk} = (z_j, w_k) \in R_{jk}$ as tag points where $z_j \in [x_{j-1}, x_j]$, $w_k \in [y_{k-1}, y_k]$,

$$= \sum_j \left(\sum_k f(z_j, w_k) \Delta y_k \right) \Delta x_j.$$

For fixed z_j,

$$\sum_k f(z_j, w_k) \Delta y_k \approx \int_c^d f(z_j, y)\, dy.$$

So, we set

$$H(x) = \int_c^d f(x, y)\, dy,$$

$$\sum_j \sum_k f(z_j, w_k) \Delta y_k \Delta x_j \approx \sum_j H(z_j) \Delta x_j$$

$$\approx \int_a^b H(x)\, dx$$

$$= \int_a^b \int_c^d f(x, y)\, dy\, dx ,$$

this is the idea behind the proof of Fubini's theorem.

Basic Property I

~ $\iint_D (\alpha f + \beta g)\, dA = \alpha \iint_D f\, dA + \beta \iint_D g\, dA$,

~ $\iint_D f\, dA \geq 0$ when $f \geq 0$ in D.

Both are clear for Riemann sums, then pass to limit.

Basic Property II (next week)